On Resolvable 2-$(14, 7, 12)$ and 3-$(14, 7, 5)$ Designs

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Abstract

We prove that there are four designs $\mathcal{F}^i$ such that each parallel class of any resolvable 2-$(14, 7, 12)$ design having at least two blocks whose intersection has six points can be constructed as a pairwise union of two parallel classes of $\mathcal{F}^l$ and $\pi(\mathcal{F}^r)$, for some $\pi \in S_{14}$. Then, with a parallel class by parallel class backtracking search we establish that there are exactly 541,192 nonisomorphic such 2-$(14, 7, 12)$ designs. We also show that there are at least the same number of 3-$(14, 7, 5)$ designs.

Keywords: *resolvable t-design; parallel class; backtracking search*

1 Introduction

A *design* is a pair $(V, \mathcal{D})$ where V is a v-set of *points* and $\mathcal{D}$ is a collection of subsets of V called *blocks*. A t-(v, k, λ) design is a design $(V, \mathcal{D})$ such that each block is of size k and each t-subset of V is contained in exactly λ blocks. It is easy to see that each point occurs in the same number r of blocks. Similarly, if b denotes the number of blocks, we have that

$$vr = bk, \qquad r\binom{k-1}{t-1} = \lambda\binom{v-1}{t-1}. \tag{1.1}$$

A design with $t = 2$ is called a *balanced incomplete block design.* For such designs, the right equation of (1.1) reduces to $r(k-1) = \lambda(v-1)$. A t-(v, k, λ) design $(V, \mathcal{D})$ is called *resolvable* if the collection $\mathcal{D}$ of blocks can be partitioned into r subsets

$$\begin{array}{llll} \{D_{11}, & D_{12}, & \ldots, & D_{1q}\}, \\ & & \vdots & \\ \{D_{r1}, & D_{r2}, & \ldots, & D_{rq}\}, \end{array}$$

called *parallel classes* of size $q = b/r$ such that every element of V appears in one block from each class.

Two designs are called *isomorphic* if there exists a permutation of the points and a permutation of the blocks that map one design onto the other. A complete enumeration of all nonisomorphic resolvable 2-$(14,7,12)$ designs that have at least two blocks whose intersection has six points is the goal of this paper. These designs have $b = 56$, $r = 26$ and $q = 2$. Before the results contained in this work only one resolvable 2-$(14,7,12)$ design was known (see [1, 3]). It was generated by the parallel classes $\{\{1,4,3,12,9,10,0\},\{2,8,6,11,5,7,\infty\}\}$ and $\{\{1,4,3,12,9,10,\infty\},\{2,8,6,11,5,7,0\}\}$ (mod 13) (see [2]).

In Section 2, via parallel classes intersection matrices [4] the resolvable 2-$(14,7,12)$ designs are classified in two types: Designs of type 1 have at least two blocks whose intersection has six points. A design is of type 2 if the intersection of any two blocks of distinct parallel classes has 2, 3, 4, or 5 points. We also prove that any resolvable 2-$(14,7,12)$ design is also a resolvable 3-$(14,7,5)$ design. Section 3 shows that there are only four resolvable designs $\mathcal{F}^i$ such that each parallel class of any design of type 1 can be constructed as a pairwise union of two parallel classes of $\mathcal{F}^l$ and $\pi(\mathcal{F}^r)$ $(l \le r)$ for some permutation $\pi \in S_{14}$, up to isomorphism. Then, in Section 4, with a parallel class by parallel class backtracking search we enumerate all designs of type 1. The computations show that there are 541,192 nonisomorphic resolvable 2-$(14,7,12)$ designs of type 1. In consequence, there are at least the same number of resolvable 3-$(14,7,5)$ designs.

2 Types of Resolvable 2-(14,7,12) Designs

In this section we will use the terminology developed in [4]. Define the *parallel classes intersection matrix* (PCIM) of two distinct parallel classes $p = \{D_{p1}, D_{p2}\}$ and $q = \{D_{q1}, D_{q2}\}$ on 14 points as the 2×2 matrix $A(p,q) = (a_{ij}(p,q))$, where $a_{ij}(p,q) = |D_{pi} \cap D_{qj}|$ ($|Z|$ denotes the cardinality of the set Z). Clearly, for each parallel class $\{D_{p1}, D_{p2}\}$ of a resolvable 2-$(14,7,12)$ design, there exist exactly 25 PCIMs $A(q,1), \ldots, A(p,p-1), A(p,p+1), \ldots, A(p,26)$. It is easy to see that for any two parallel classes p and q of a resolvable 2-$(14,7,12)$ design, $a_{i1}(p,q) + a_{i2}(p,q) = 7$ for $i = 1,2$. Hence for any $1 \le i \le 2$, the set $\{a_{i1}(p,q), a_{i2}(p,q)\}$ is one of the sets $\{7,0\}, \{6,1\}, \{5,2\}$ or $\{4,3\}$. This means that

$$a_{i1}(p,q)(a_{i1}(p,q)-1) + a_{i2}(p,q)(a_{i2}(p,q)-1) \ge 18.$$

On the other hand, there are 252 occurrences of two points from D_{pi} in the same block (21 in D_{pi} and 231 in the other blocks). Thus, for any class p,

$$\frac{1}{2}\sum_{\substack{q=1\\q\neq p}}^{26}\sum_{j=1}^{2} a_{ij}(p,q)(a_{ij}(p,q)-1) = 231. \tag{2.1}$$

Hence, each entry of any PCIM is less than 7; that is, there are no repeated blocks. Therefore, each column or row of any PCIM $A(p,q)$, up to permutations, has the form:

$$[6,1], \quad [5,2] \quad \text{or} \quad [4,3]. \tag{2.2}$$

Let D_{pi} be a block of a resolvable 2-$(14,7,12)$ design. Then, solving (2.1), the ith rows of the matrices $A(p,1),\ldots,A(p,p-1),A(p,p+1),\ldots,A(p,26)$ have either the pattern:

$$[6,1],\ \underbrace{[4,3],\ldots,[4,3]}_{24}, \tag{2.3}$$

or

$$[5,2],\ [5,2],\ [5,2],\ \underbrace{[4,3],\ldots,[4,3]}_{22}. \tag{2.4}$$

It follows from (2.2) that for any two distinct parallel classes p and q of a resolvable 2-$(14,7,12)$ design, there exist exactly, up to exchange of rows or columns, three possible PCIMs:

$$\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}, \quad \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}.$$

We say that these parallel classes p and q meet in *type* I, II or III, respectively. Note that if the matrix $A(p,q)$ is of type I, II or III, then the parallel class $\{D_{p1},D_{p2}\}$ has, respectively, 30, 22 or 18 occurrences of two points from the parallel class $\{D_{q1},D_{q2}\}$. We define the sets $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ of all PCIMs of type I, II and III, respectively. For each parallel class p of a resolvable design and $1 \leq h \leq 3$, we define $f(p,h)$ as the number of parallel classes q in the design such that $A(p,q) \in \mathcal{T}_h$.

Let $\{D_{p1},D_{p2}\}$ be a parallel class of a resolvable 2-$(14,7,12)$ design. Let x_1, x_2 and x_3 be the number of matrices from the 26 PCIMs $A(p,1),\ldots,A(p,p-1),A(p,p+1),\ldots,A(p,26)$ of type I, II or III, respectively. It follows that

$$\begin{aligned} x_1 + x_2 + x_3 &= 25, \\ 30x_1 + 22x_2 + 18x_3 &= 231. \end{aligned}$$

These equations have two solutions: $x_1 = 1$, $x_2 = 0$, $x_3 = 24$, and $x_1 = 0$, $x_2 = 3$, $x_3 = 22$. So we have a result similar to [4, Thereom 2.1]:

Theorem 2.1 *A necessary condition for the existence of a resolvable* 2-(14, 7, 12) *design is that for each parallel class p, the* 25 *PCIMs* $A(p, 1), \ldots, A(p, p-1), A(p, p+1), \ldots, A(p, 26)$ *have one of the following patterns:*

P.1. *One matrix of type I and* 24 *matrices of type III,*

P.2. *Three matrices of type II and* 22 *matrices of type III.*

It follows from Theorem 2.1 that there are two types of resolvable 2-(14, 7, 12) designs: Designs of type 1 have at least one pair of parallel classes such that they have pattern P.1. A design is of type 2 if all parallel classes of the design have the pattern P.2. Clearly, a design is of type 1 if and only if the design has at least two blocks whose intersection has six points.

Finally, we prove the following theorem:

Theorem 2.2 *Let* $k \geq 3$. *Then every resolvable* 2-$(2k, k, \lambda)$ *design is also a resolvable* 3-$(2k, k, \lambda')$ *design, where* $\lambda' = \frac{\lambda(k-2)}{2(k-1)}$. *In particular, for* $k = 7$ *and* $\lambda = 12$ *we have* $\lambda' = 5$.

Proof. Each parallel class of every resolvable 2-$(2k, k, \lambda)$ design has two blocks. We will prove that every 3-subset of V is contained in exactly λ' blocks. For any three points of V we have two possibilities in each parallel class of the design: either (1) the three points occur together in the same block, or (2) two points belong to one block and the other point belongs to the other block. Let x_1 be the number of classes satisfying (1) and let x_2 be the number of classes satisfying (2). Since our design is a resolvable 2-$(2k, k, \lambda)$ design,

$$\begin{aligned} x_1 + x_2 &= r, \\ 3x_1 + x_2 &= 3\lambda. \end{aligned}$$

Solving these equations and using (1.1), we obtain $x_1 = \lambda' = \frac{\lambda(k-2)}{2(k-1)}$. (Clearly, λ' is a positive integer whenever a resolvable 2-$(2k, k, \lambda)$ design exists.) □

3 Four Resolvable Designs

In this section, X, Y and V denote respectively the sets $\{0, 1, 2, 3, 4, 5\}$, $\{8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$ and $X \cup \{6, 7\} \cup Y$, and π denotes the permutation (08)(19)(2A)(3B)(4C)(5D). Note that $\pi(X) = Y$ and $\pi(Y) = X$. For each subset Z of V and each parallel class p we define the vector

$$n(p, Z) = [|D_{p1} \cap Z|, |D_{p2} \cap Z|].$$

Clearly, for each subset of V, there are 26 such vectors, one for each parallel class.

Let $(V, \mathcal{D})$ be a resolvable 2-$(14, 7, 12)$ design and let Z a subset of V. Define

$$\mathcal{D} \cap Z = \{D \cap Z \mid D \in \mathcal{D}\}.$$

Thus the pair $(V \cap Z, \mathcal{D} \cap Z)$ is a design. Finally, for each two pairs of sets (D_1, D_2) and (F_1, F_2), define

$$(D_1, D_2) \oplus (F_1, F_2) = (D_1 \cup F_1, D_2 \cup F_2).$$

In this paper, we only construct all nonisomorphic resolvable 2-$(14, 7, 12)$ designs that has at least two blocks whose intersection has six points exists. Then, we can assume without loss of generality that the first and the second parallel classes meet in type I; that is, the PCIM $A(1, 2)$ is of type I. Thus, again without loss of generality the first and second parallel classes will be taken as $\{0123456, 789\mathrm{ABCD}\}$ and $\{0123457, 689\mathrm{ABCD}\}$, respectively. Note that the other parallel classes of the design meet with the first and second parallel classes in type III.

Suppose that a $(V, \mathcal{D})$ resolvable 2-$(14, 7, 12)$ design of type 1. Then, there are 180 occurrences of two points from X in the same block (30 in blocks of the first two parallel classes and 150 in blocks of the other classes). Since the PCIMs $A(1, q)$ and $A(2, q)$ are of type III, we have that for each class $q \notin \{1, 2\}$, the vector $n(q, X)$, up to permutations, has either the form:

$$[4, 2] \quad \text{or} \quad [3, 3].$$

Let x_1 and x_2 be the number of vectors of the form $[4, 2]$ and $[3, 3]$, respectively. So

$$\begin{aligned} x_1 + x_2 &= 24, \\ 7x_1 + 6x_2 &= 150. \end{aligned}$$

The solution of the above equations is $x_1 = 6$, $x_2 = 18$. Therefore, the parallel classes of the $(V, \mathcal{D})$ design can be ordered in such a way that for any class $q \in \{3, \ldots, 8\}$ the vector $n(q, X)$ has the form $[4, 2]$, and for any $q \in \{9, \ldots, 26\}$ the vector $n(q, X)$ has the form $[3, 3]$. (That is, $|D_{q1} \cap X| = 4$ for $q = 3, \ldots, 8$ and $|D_{q1} \cap X| = 3$ for $q = 9, \ldots, 26$.) Similarly, we can assume that for any parallel class $q \in \{9, \ldots, 14\}$ the vector $n(q, Y)$ has the form $[4, 2]$. Henceforth the parallel classes of any resolvable 2-$(14, 7, 12)$ of type 1 will be ordered following this form. According to this ordering of the parallel classes, we define the following collection of subsets of X:

$$\mathcal{D}(3, 8, X, j) = \{D_{3j} \cap X, \ldots, D_{8j} \cap X\},$$

where D_{ij} is the jth block of the ith parallel class of the design. Clearly, the pair $(V \cap X, \mathcal{D}(3, 8, X, j))$ is a design.

Lemma 3.1 *The design $\mathcal{D}(3,8,X,2)$ determines completely the design $\mathcal{D} \cap X$.*

Proof. Clearly, $D_{i1} \cap X = X - (D_{i2} \cap X)$ for $i = 3, \ldots, 8$. So $\mathcal{D}(3,8,X,2)$ determines $\mathcal{D}(3,8,X,1)$. The remaining classes of $\mathcal{D} \cap X$ are determined by $\mathcal{D}(3,8,X,1)$, in such a way that every 3-subset of X is contained in exactly 5 blocks of the design (see Table 1). □

Lemma 3.2 *Each 5-subset of X contains exactly two blocks from $\mathcal{D}(3,8,X,1)$.*

Proof. Let Q be a 5-subset of X. Let x_1, x_2 and x_3, be the number of parallel classes $p \in \{3, \ldots, 26\}$ such that $n(p,Q)$ is $[5,0]$, $[4,1]$ or $[3,2]$, respectively, up to component permutations. By Theorem 2.2 each one of the 10 triples of Q is contained in exactly five blocks. Since each 3-subset of Q occurs in the first and second parallel classes, there are 30 occurrences of 3-subsets of Q in the other parallel classes. So

$$\begin{aligned} x_1 + x_2 + x_3 &= 24, \\ 10x_1 + 4x_2 + x_3 &= 30. \end{aligned}$$

The only solution of this system is $x_1 = 0$, $x_2 = 2$, $x_3 = 22$. Thus, there are two parallel classes $p_1, p_2 \in \{3, \ldots, 8\}$ such that $n(p_1, Q) = n(p_2, Q) = [4,1]$. Therefore, $\mathcal{D}(3,8,X,1)$ has exactly two 4-subsets of Q. □

Lemma 3.3 *Each point of X belongs to exactly two blocks from $\mathcal{D}(3,8,X,2)$.*

Proof. Clearly, for each $x \in X$, $x \in D_{i2} \cap X$ if and only if $D_{i1} \cap X \subseteq X - \{x\}$. By Lemma 3.2, the 5-subset $X - \{x\}$ contains exactly two blocks of $\mathcal{D}(3,8,X,1)$. So x belongs to two blocks of $\mathcal{D}(3,8,X,2)$. □

Theorem 3.4 *For any $(V,\mathcal{D})$ resolvable 2-$(14,7,12)$ design that has at least two blocks whose intersection has six points we have*

1. *$\mathcal{D} \cap X$ is isomorphic to one of $\mathcal{F}^1$, $\mathcal{F}^2$, $\mathcal{F}^3$ and $\mathcal{F}^4$ given in* Table 1*;*
2. *$\mathcal{D} \cap Y$ is isomorphic to one of $\pi(\mathcal{F}^1)$, $\pi(\mathcal{F}^2)$, $\pi(\mathcal{F}^3)$ and $\pi(\mathcal{F}^4)$.*

Proof. By Lemma 3.1 it suffices to determine the design $\mathcal{D}(3,8,X,2)$. In several places we will use the fact that the first two parallel classes of the design $\mathcal{D}$ are invariant under any permutation on X. Thus, without loss of generality it can be assumed that $D_{32} \cap X = 45$, so $D_{31} \cap X = 0123$. By Lemma 3.3, every point of X occurs in exactly two blocks of $\mathcal{D}(3,8,X,2)$. So there are three cases to consider:

(1) The design $\mathcal{D}(3,8,X,2)$ has only one block that occurs twice. Thus we can assume $D_{42} \cap X = 45$, so $D_{41} \cap X = 0123$. Since the first four parallel classes of

$\mathcal{D}$ are invariant under any permutation on $\{0,1,2,3\}$, $D_{52} \cap X = 23$. It is not hard to see that $D_{62} \cap X = 13$, $D_{72} \cap X = 02$ and $D_{82} \cap X = 01$. In this case we have $\rho(\mathcal{D} \cap X) = \mathcal{F}^2$ for some permutation ρ on X.

(2) The design $\mathcal{D}(3,8,X,2)$ has at least two blocks that occurs twice. Then, we can assume without of generality that $D_{42} \cap X = 45$, $D_{52} \cap X = D_{62} \cap X = 23$ and $D_{72} \cap X = D_{82} \cap X = 01$. In this case we have $\rho(\mathcal{D} \cap X) = \mathcal{F}^3$ for some permutation ρ on X.

(3) The design $\mathcal{D}(3,8,X,2)$ has no repeated blocks. Since the first three parallel classes of $\mathcal{D}$ are invariant under any permutations on $\{0,1,2,3\}$ and the permutation (45), we can assume that $D_{42} \cap X = 35$, so $D_{41} \cap X = 0124$. Any permutation on $\{0,1,2\}$ and the permutation (34) leave invariant the first four class. Hence, we have three possibilities: $D_{52} \cap X = 34$, 24 or 12. Suppose that $D_{52} \cap X = 34$. Then, by Lemma 3.3, $D_{62} \cap X = 12$, $D_{72} \cap X = 02$ and $D_{82} \cap X = 01$. Clearly the third possibility leads to the same design. For the remaining possibility, $D_{52} \cap X = 24$, we have $D_{62} \cap X = 13$, because $D_{62} \cap X = 23$ implies that $D_{72} \cap X = D_{82} \cap X = 01$, which gives a contradiction (the design has no repeated blocks). It follows from Lemma 3.3 that $D_{72} \cap X = 02$ and $D_{82} \cap X = 01$. Therefore, we have $\rho(\mathcal{D} \cap X) = \mathcal{F}^4$ or $\rho(\mathcal{D} \cap X) = \mathcal{F}^1$ for some permutation ρ on X. This completes the proof of the first part of the theorem.

Since the permutations on X leave invariant the set Y, and $\pi(X) = Y$ and $\pi(Y) = X$, the first part of the theorem implies the second part. $\square$

Table 1.

$\mathcal{F}^1$		$\mathcal{F}^2$		$\mathcal{F}^3$		$\mathcal{F}^4$	
012345		012345		012345		012345	
012345		012345		012345		012345	
0123	45	0123	45	0123	45	0123	45
0124	35	0123	45	0123	45	0124	35
0135	24	0145	23	0145	23	0125	34
0245	13	0245	13	0145	23	0345	12
1345	02	1345	02	2345	01	1345	02
2345	01	2345	01	2345	01	2345	01
012	345	012	345	012	345	013	245
013	245	013	245	013	245	013	245
014	235	014	235	014	235	014	235
014	235	014	235	015	234	014	235
015	234	015	234	023	145	015	234
015	234	015	234	024	135	015	234
023	145	023	145	024	135	023	145
023	145	024	135	024	135	023	145
024	135	024	135	025	134	024	135
025	134	025	134	025	134	024	135
025	134	025	134	025	134	025	134
034	125	034	125	034	125	025	134
034	125	034	125	034	125	034	125
034	125	034	125	034	125	034	125
035	124	035	124	035	124	035	124
035	124	035	124	035	124	035	124
045	123	035	124	035	124	045	123
045	123	045	123	045	123	045	123

Theorem 3.5 *For any $(V, \mathcal{D})$ resolvable 2-$(14, 7, 12)$ design which has at least two blocks whose intersection has six points, there exist a permutation ρ on the set V and integers $1 \le l \le r \le 4$ satisfying:*

1. *For $3 \le i \le 8$,*

$$(\rho(D_{i1}), \rho(D_{i2})) = (F^l_{i1}, F^l_{i2}) \oplus (\pi(F^r_{sj_1}), \pi(F^r_{sj_2})) \oplus (\emptyset, \{6, 7\}),$$

for some $9 \le s \le 26$ and $1 \le j_1 \ne j_2 \le 2$.

2. *For $9 \le i \le 14$,*

$$(\rho(D_{i1}), \rho(D_{i2})) = (F^l_{sj_1}, F^l_{sj_2}) \oplus (\pi(F^r_{t1}), \pi(F^r_{t2})) \oplus (\emptyset, \{6, 7\}),$$

for some $9 \le s \le 26$, $t = i - 6$ and $1 \le j_1 \ne j_2 \le 2$.

3. *For* $15 \leq i \leq 26$,

$$(\rho(D_{i1}), \rho(D_{i2})) = (F^l_{s1}, F^l_{s2}) \oplus (\pi(F^r_{tj_1}), \pi(F^r_{tj_2})) \oplus (\{h_1\}, \{h_2\}),$$

for some $9 \leq s \leq 26$, $9 \leq t \leq 26$, $1 \leq j_1 \neq j_2 \leq 2$ *and* $6 \leq h_1 \neq h_2 \leq 7$.

Proof. By Theorem 3.4, there exist two permutations ρ_1 on X and ρ_2 on Y that are isomorphisms between $(V \cap X, \mathcal{D} \cap X)$ and $\mathcal{F}^l$, and between $(V \cap Y, \mathcal{D} \cap Y)$ and $\pi(\mathcal{F}^r)$ $(1 \leq l, r \leq 4)$, respectively. If $l > r$, we take $\pi(V, \mathcal{D})$ instead of $(V, \mathcal{D})$, so we can assume that $l \leq r$.

Define $\rho(z) = \rho_1(z)$ if $z \in X$, $\rho(z) = \rho_2(z)$ if $z \in Y$ and $\rho(z) = z$ otherwise. It is not hard to check that the permutation ρ satisfies (1–3). □

4 The construction

In this section, we will use the Theorem 3.5 to construct all resolvable 2-$(14, 7, 12)$ designs having at least two blocks whose intersection has six points in a parallel class by parallel class backtracking search. Our initial solution $\mathcal{D}$ is the design with the first two parallel classes $\{0123456, 789\text{ABCD}\}$ and $\{0123457, 689\text{ABCD}\}$. Clearly, any permutation on X or on Y leaves invariant these two classes. A class p can be chosen as a parallel class of $\mathcal{D}$, if $\mathcal{D} \cup \{p\}$ is a partial 2-$(14, 7, 12)$ design and also a partial 3-$(14, 7, 5)$ design. Moreover, each parallel class of the design $\mathcal{D} \cup \{p\}$ partially satisfies Theorem 2.1. This means that for each parallel class q in $\mathcal{D} \cup \{p\}$ we have either

1. $f(q, 1) \leq 1$, $f(q, 2) = 0$ and $f(q, 3) \leq 24$, or
2. $f(q, 1) = 0$, $f(q, 2) \leq 3$ and $f(q, 3) \leq 22$.

For each $1 \leq i \leq 4$, we define $\mathcal{J}^i$ as the set of all distinct parallel classes from the 9th to the 26th of the design $\mathcal{F}^i$. Also, for each parallel class $p \in \mathcal{J}^i$, we define $m(p)$ as the number of occurrences of p in the design $\mathcal{F}^i$.

Let G_1, G_2, G_3 and G_4 be the groups generated by the permutation sets

$$\{(89)(\text{AB})(\text{CD}), (9\text{A})(\text{BC})\}, \quad \{(\text{CD}), (89)(\text{AB}), (9\text{A})\},$$
$$\{(89), (\text{AB}), (\text{CD}), (8\text{A})(9\text{B}), (\text{AC})(\text{BD})\} \quad \text{and}$$
$$\{(89), (9\text{A}), (\text{BC}), (\text{CD}), (8\text{B})(9\text{C})(\text{AD})\},$$

respectively. It is not hard to check that for $1 \leq r \leq 4$, the group G_r leaves invariant the design $\pi(\mathcal{F}^r)$ (with possible reordering of the parallel classes and reordering of the blocks in the same parallel class). Clearly, the group G_r $(1 \leq r \leq 4)$ fixes each parallel class of the design $\mathcal{F}^l$ $(1 \leq l \leq 4)$.

For $1 \leq r \leq 4$, the set $\{(\pi(F^r_{sj_1}), \pi(F^r_{sj_2})) \mid s \in \mathcal{J}^r,\ (j_1, j_2) = (1,2),(2,1)\}$ is partitioned into orbits under the action of the group G_r. Then we define $\mathcal{H}^r$ as the set that contains exactly the leader of each orbit. So by a simple calculation we have

$$
\begin{aligned}
\mathcal{H}^1 &= \{(89\mathrm{A}, \mathrm{BCD}), (89\mathrm{C}, \mathrm{ABD}), (8\mathrm{BC}, 9\mathrm{AD})\},\\
\mathcal{H}^2 &= \{(89\mathrm{A}, \mathrm{BCD}), (89\mathrm{C}, \mathrm{ABD}), (8\mathrm{BC}, 9\mathrm{AD}), (8\mathrm{CD}, 9\mathrm{AB})\},\\
\mathcal{H}^3 &= \{(89\mathrm{A}, \mathrm{BCD}), (8\mathrm{AC}, 9\mathrm{BD})\},\\
\mathcal{H}^4 &= \{(89\mathrm{B}, \mathrm{ACD})\}.
\end{aligned}
$$

In order to generate all nonisomorphic resolvable 2-$(14,7,12)$ designs of type 1, by Theorem 3.5 it suffices to consider only ten cases: (l, r), $1 \leq l \leq r \leq 4$. Then for each one (l, r) of these ordered pairs, the ith parallel class will be chosen in the following sets:

For $i = 3$,

$$\{(F^l_{31}, F^l_{32}) \oplus (H_1, H_2)) \oplus (\emptyset, \{6,7\}) \mid (H_1, H_2) \in \mathcal{H}^r\}.$$

For $4 \leq i \leq 8$,

$$\{(F^l_{i1}, F^l_{i2}) \oplus (\pi(F^r_{sj_1}), \pi(F^r_{sj_2})) \oplus (\emptyset, \{6,7\}) \mid s \in \mathcal{J}^r,\ (j_1, j_2) = (1,2),(2,1)\}.$$

For $9 \leq i \leq 14$,

$$\{(F^l_{sj_1}, F^l_{sj_2}) \oplus (\pi(F^r_{t1}), \pi(F^r_{t2})) \oplus (\emptyset, \{6,7\}) \mid s \in \mathcal{J}^l,\ t = i-6,\ (j_1, j_2) = (1,2),(2,1)\}.$$

For $15 \leq i \leq 26$,

$$
\begin{aligned}
\{(F^l_{s1}, F^l_{s2}) \oplus (\pi(F^r_{tj_1}), \pi(F^r_{tj_2})) \oplus (\{h_1\}, \{h_2\}) \mid s \in \mathcal{J}^l,\ t \in \mathcal{J}^r,&\\
(j_1, j_2) = (1,2),(2,1),\ (h_1, h_2) = (6,7),(7,6)\}.&
\end{aligned}
$$

For each choice of the third parallel class, we found the group P_r of permutations on Y that leaves invariant the design $\pi(\mathcal{F}^r)$ and fixes the third class. Clearly, the group P_r fixes each parallel class of the designs $\mathcal{F}^l$. Then the set

$$\{(\pi(F^r_{sj_1}), \pi(F^r_{sj_2})) \mid s \in \mathcal{J}^r,\ (j_1, j_2) = (1,2),(2,1)\}$$

is partitioned into orbits under the action of the group P_r. To generate all nonisomorphic designs it suffices to choose the fourth parallel class in the set $\mathcal{P}_r$, which contains exactly the leader of each orbit.

At each level of the backtrack algorithm, the class $q = (\pi(F^r_{sj_1}), \pi(F^r_{sj_2}))$ of $\pi(F^r)$ is chosen as a part of a parallel class of the partial design $\mathcal{D}$ whenever its number of occurrences in $\mathcal{D}$ is less than or equal to $m(q)$.

Note that any two designs constructed respectively via the designs $(\mathcal{F}^i, \pi(\mathcal{F}^j))$ and $(\mathcal{F}^l, \pi(\mathcal{F}^r))$, may only be isomorphic if $(i,j) = (l,r)$. Hence, as we encounter a design constructed via $(\mathcal{F}^l, \pi(\mathcal{F}^r))$ we test for isomorphism with any of the designs recorded so far, constructed also via $(\mathcal{F}^l, \pi(\mathcal{F}^r))$. Then, if the new design is nonisomorphic to all such designs, it is recorded. To check for isomorphism we used the nauty program [5]. A total of 541,192 nonisomorphic resolvable 2-$(14,7,12)$ designs of type 1 were found. The size of the automorphism groups are presented in Table 2. Columns (l,r) give the number of nonisomorphic designs found via the designs $(\mathcal{F}^l, \pi(\mathcal{F}^r))$. The CPU time employed to run the ten cases were about 501 hours on a 800 MHz PC computer.

Table 2. The nonisomorphic resolvable 2-$(14,7,12)$ designs of type 1

	(l,r)										
$\lvert\mathrm{Aut}(D)\rvert$	(1,1)	(1,2)	(1,3)	(1,4)	(2,2)	(2,3)	(2,4)	(3,3)	(3,4)	(4,4)	
1	255014	150038	4823	70608	27142	3065	22839	162	1298	5070	
2	181	63	1	97	161	24	52	4	48	171	
3	56	25	11	70	9	1	4	2	24	31	
4	5	2	1	3	10	0	2	1	1	14	
6	7	0	1	12	2	0	0	3	4	13	
8	1	0	0	0	0	0	0	0	0	0	
12	2	0	0	2	1	0	1	1	1	5	
13	1	0	0	0	0	0	0	0	0	0	
24	1	0	0	0	0	0	0	0	0	0	
156	0	0	0	0	0	0	0	0	0	1	
Total	255268	150128	4837	70792	27325	3090	22898	173	1376	5305	541192
CPU (hr)	281.7	79.2	3.2	29.3	49.1	3.3	26.4	0.5	3	25.8	501.5

Finally, Table 3(a) and (b) give the two resolvable 2-$(14,7,12)$ designs of type 1 whose automorphism groups have sizes 24 and 156, respectively. For each parallel class p of the design (b) the PCIMs $A(p,1), \ldots, A(p,p-1), A(p,p+1), \ldots, A(p,26)$ have the pattern P.1 of Theorem 2.1. Moreover, this design and the resolvable 2-$(14,7,12)$ design generated by the parallel classes given in Section 1 are isomorphic.

Table 3. The resolvable 2-$(14, 7, 12)$ designs of type 1 (a) and (b) with automorphism group of size 24 and 156, respectively

(a)		(b)	
0123456	789ABCD	0123456	789ABCD
0123457	689ABCD	0123457	689ABCD
012389C	4567ABD	012389B	4567ACD
01248AD	35679BC	01248AC	35679BD
01359AB	24678CD	01259AD	34678BC
0245ABC	136789D	03458BD	12679AC
13458BD	02679AC	13459CD	02678AB
23459CD	01678AB	2345ABC	016789D
01489AB	23567CD	14589AB	02367CD
02589AC	13467BD	03589AC	12467BD
12389BD	04567AC	23489AD	01567BC
2348ACD	015679B	1258BCD	034679A
0359BCD	124678A	0249BCD	135678A
145ABCD	0236789	013ABCD	2456789
0126BCD	345789A	0136ABD	245789C
0137ACD	245689B	01468AC	23579BD
01469CD	23578AB	014789D	2356ABC
01578CD	23469AB	01579BC	23468AD
0236ABD	145789C	02378CD	14569AB
02479BD	13568AC	02469BC	13578AD
02578BD	13469AC	02569AD	13478BC
03468BC	12579AD	02578AB	13469CD
03478BC	12569AD	03479AB	12568CD
03479AD	12568BC	035689C	1247ABD
03568AD	12479BC	04568BD	12379AC
045689D	1237ABC	0457ACD	123689B

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